Mathematical modeling of triangle-mesh-modeled three-dimensional surface objects for digital holography

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We develop a mathematical model of triangle-mesh-modeled three-dimensional (3D) surface objects for digital holography. The proposed mathematical model includes the analytic angular spectrum representation of image light fields emitted from 3D surface objects with occlusion and the computation method for the developed light field representation. Reconstruction of computer-generated holograms synthesized by using the developed model is demonstrated experimentally. © 2008 Optical Society of America OCIS codes: 090.1760, 090.2870, 090.1970.

1. Introduction

Digital holography is generally referred to as the study of the analysis and synthesis of a light field by means of digital computers and processors [1-3]. Much emphasis is placed on computer-generated holograms (CGHs) among several areas of digital holography. It has been perceived as a promising technology for generating three-dimensional (3D) images and continues to be intensively researched [4,5]. From the theoretical point of view, the generation of 3D image light fields requires both phase and amplitude modulations of light fields. In principle, the digital holography technology of CGHs provides the only method to generate real physical image light fields of mathematically modeled 3D objects or real 3D objects by controlling both the phase and the amplitude information of light fields.

However, at present, in the field of 3D display it is commonly understood that it is difficult to realize practical 3D display systems based on CGH because of several practical limitations in spite of the fact that CGH is the only way to produce true holographic 3D images. Representative examples of limitations include the huge computational cost required when obtaining CGH patterns by existing CGH algorithms and the impractically large and high-resolution spatial light modulators (SLMs) needed for generating realistic 3D holographic images [6]. The realization of practical CGH-based 3D displays is still a challenging goal requiring further intense research.

Regarding the computational problem, more efficient and accurate algorithms for synthesizing CGHs are continuously sought [7–11]. In the general context of digital holography as well as CGH synthesis, the mathematical modeling of image light fields emitted from 3D objects that describe the 3D objects is one of the most important issues. In addition, the computational modeling of optical transformations of light fields, such as those propagating through paraxial or nonparaxial optical systems, and human perception of 3D images through eyes are important topics.

For the past decade, point-source techniques such as coherent ray tracing [4,5] have been widely used to compute light fields of 3D objects. Many sampled point sources are uniformly deposited on an object's surface. The respective optical field generated from each sampling point source is superposed to generate

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the collective 3D image light field. Wave-oriented methods calculate light fields emitted from objects defined by planar segments [9,10,12,13], using the fast Fourier transform or analytic wave optic formulations for calculating the complex light field. These wave optic approaches lead to better computational efficiency than in the point-source-based methods. Additionally, the edge sharpness of polyhedron objects, texture, and shade effects can be more accurately represented by the wave-oriented methods than by the point-source-based methods.

In this paper, a mathematical model describing 3D image light fields emitted from arbitrary 3D objects represented by triangle meshes is developed. Within the mathematical model established in this paper, a triangular facet becomes a computational unit. Recently, in [9], the angular spectrum representation of 3D surface objects with shade and texture was addressed. In [9], the remapping of the angular spectrum that is needed in the calculation of the light field radiated from a tilted planar facet is accomplished by a simple interpolation of the sampled angular spectrum.

The main features of the described mathematical model are as follows. First, the analytic angular spectrum representation formulation of the light field emitted from a triangle-mesh-modeled 3D object is derived, which gives an exact angular spectrum representation without the interpolated remapping adopted in the previous study [9]. Second, a simple geometric modeling of the occlusion effect of any 3D surface objects is provided through the geometric observable facet selection method by ray tracing. In addition, a diffusive surface model that is feasible for producing several surface texture and shade effects [9] is devised. Full details are described in the main part of this paper.

In Section 2, the analytic angular spectrum representation of the light field emitted by a unit triangular facet with surface diffusiveness is derived. In Section 3 we describe the construction of the collective light field of a 3D object with occlusion by summing up the radiation field generated from each triangular facet with the observable facet selection algorithm. In Section 4, numerical and experimental results of reconstruction of CGHs synthesized with the developed mathematical model are presented for comparison. In Section 5, the concluding remarks are given.

2. Light Field Radiated from a Triangular Facet with Surface Diffusiveness

In this section, a triangle-mesh-based modeling of a 3D surface object and a mathematical representation of the light field radiated from a unit triangular facet with surface diffusiveness are elucidated.

In many commercial 3D graphics programs, 3D objects are modeled by a triangle-mesh structure. A triangle-mesh-modeling example of two hands composed of 2436 triangles is presented in Fig. 1(a).

Adopting this triangle-mesh-based object modeling, we place our focus on the mathematical representation of the image light field radiated from the meshed surfaces of 3D objects. This problem is closely related to the interesting inverse problem in diffractive optics of forming specific light intensity distributions on arbitrary curved surfaces represented by triangle meshes as well as to CGH synthesis and reconstruction.

The first step of building the proposed model is the derivation of the angular spectrum representation of light field radiated from a tilted triangular facet. In Fig. 1, the triangle-mesh-modeling concept and coordinate system convention are illustrated. Let the coordinate system shown in Figs. 1(a) and 1(b) be termed the global coordinate system.

Let us consider the *k*th unit triangle in the triangle-mesh-modeled object shown in Fig. 1(b). The local coordinate system on the *k*th unit triangle is defined as shown in Fig. 1(c). Let the center of mass of the triangular facet and the outer normal vector \mathbf{n}_k be set to the origin of the local coordinate system and the z' axis, respectively, and then, by the right-hand rule, the local coordinate system (x', y', z') is constructed as shown in Fig. 1(c). Figure 1(d) illustrates the relationship between the local coordinate system,



Fig. 1. (Color online) (a) Triangle mesh modeling of 3D object (two hands). (b) Triangle mesh surface-polyhedron objects in the global coordinate system. (c) Local coordinate system of the kth unit triangle. (d) The relationship between the local coordinate system and the global coordinate system.

where (x_k, y_k, z_k) is the center of mass of the *k*th unit triangle.

The *k*th unit triangle is placed on an infinite plane

$$a_k x + b_k y + c_k z + d_k = 0,$$
 (1a)

where the vector components of the normal vector of the unit triangle \mathbf{n}_k , (a_k, b_k, c_k) , and d_k are given by, respectively,

$$(a_k, b_k, c_k) = (\cos \phi_k \sin \theta_k, \sin \phi_k \sin \theta_k, \cos \theta_k), \quad (1b)$$

$$d_k = -a_k x_k - b_k y_k - c_k z_k \,. \tag{1c}$$

The spatial coordinate variables (x', y', z') in the local coordinate system correspond to the spatial coordinate variables (x, y, z) in the global coordinate system by the rotation transform

$$\begin{pmatrix} x'\\ y'\\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta_k \cos\phi_k & \cos\theta_k \sin\phi_k & -\sin\theta_k\\ -\sin\phi_k & \cos\phi_k & 0\\ \sin\theta_k \cos\phi_k & \sin\theta_k \sin\phi_k & \cos\theta_k \end{pmatrix} \times \begin{pmatrix} x - x_k\\ y - y_k\\ z - zk \end{pmatrix}.$$
 (2)

The point (x_k, y_k, z_k) in the global coordinate system corresponds to the origin of the local coordinate system. Then the triangular facet is placed on the x'y'plane (z' = 0). Let us assume that the kth unit triangle is a transparent small aperture on an infinite opaque screen and that a carrier plane wave $\eta_0 \exp(j2\pi(\alpha_0 x + \beta_0 y + \gamma_0 z))$ is incident on the unit triangle aperture in the global coordinate system. The unit triangle aperture diffracts the carrier plane wave and generates a complex diffraction field $W_k(x,y,z).$

In the local coordinate system, the illuminating plane wave is expressed as $\eta_0 \exp(j2\pi(\alpha'_0(x'+x'_k)+$ $\beta'_0(y'+y'_k)+\gamma'_0(z'+z'_k)))$ from Eq. (2). Let us denote the generated diffraction field in the local coordinate system by $W_k(x', y', z')$. The light field distribution on the kth unit triangular facet is represented by the angular spectrum representation in the local coordinate system

$$\begin{split} W_{k}(x',y',0) &= \eta_{0} \exp(j2\pi(\alpha'_{0}(x'+x'_{k})+\beta'_{0}(y'+y'_{k}) \\ &+ \gamma'_{0}z'_{k}))U_{k}(x',y') \\ &= \eta_{0} \exp(j2\pi(\alpha'_{0}(x'+x'_{k})+\beta'_{0}(y'+y'_{k}) \\ &+ \gamma'_{0}z'_{k}))\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A_{k}(\alpha',\beta')\exp(j2\pi(\alpha'x' \\ &+ \beta'y'))\mathrm{d}\alpha'\mathrm{d}\beta', \end{split}$$
(3a)

. . . .

where $U_k(x', y')$ is the transmittance function of the *k*th unit triangular facet and $A_k(\alpha', \beta')$ is its analytic 2D Fourier transform. The above expression can be manipulated as

$$egin{aligned} W_k(x',y',0) &= \int_{-\infty}^\infty \int_{-\infty}^\infty \eta_0 \exp(j2\pi(lpha_0' x_k'+eta_0' y_k' \ &+ \gamma_0' z_k')) A_k(lpha'-lpha_0',eta' \ &- eta_0') \exp(j2\pi(lpha' x'+eta' y')) \mathrm{d}lpha' \mathrm{d}eta'. \end{aligned}$$

The light field distribution emitted (diffracted) from the kth triangular facet in whole space, $W_k(x', y', z')$, is given by

$$egin{aligned} W_k(x',y',z') &= \int_{-\infty}^\infty \int_{-\infty}^\infty \eta_0 \exp(j2\pi(lpha_0' x_k'+eta_0' y_k' \ &+ \gamma_0' z_k')) A_k(lpha'-lpha_0',eta_0' \ &- eta_0') \exp(j2\pi(lpha' x'+eta' y'+\gamma' z')) \mathrm{d}lpha' \mathrm{d}eta', \end{aligned}$$

where γ' is defined by $\gamma' = \sqrt{(1/\lambda)^2 - \alpha'^2 - \beta'^2}$ and λ is the free space wavelength of the radiation light field. Equation (3c) is the angular spectrum representation of $W_k(x', y', z')$, where $(\alpha', \beta', \gamma')$ is the Fourier spatialfrequency vector.

Here, the analytic diffusive surface model is introduced. The diffusive surface model is necessary to create and control the wide viewing angle of the unit triangular facet. Let us consider the kth triangular facet $P_1P_2P_3$ in its local coordinate system as shown in Fig. 2. Uniformly sampled M - 1 points are taken on the lines P_3P_1 and P_2P_3 so that the lines P_3P_1 and P_2P_3 are divided by M parts. As a result, we can divide the facet into M(M+1)/2 similar upwardoriented-triangles denoted $\Lambda_{k,\uparrow}$ and M(M-1)/2 similar downward-oriented triangles denoted $\Lambda_{k,\downarrow}$ as shown in Figs. 2(a) and 2(b), respectively. Mathematically, the transmittance function of the diffusive surface $U_k(x', y')$ is represented by the sum of the partial transmittance function of the upwardoriented-triangle area, $U_{k,\uparrow}(x',y')$, and that of the downward-oriented-triangle area, $U_{k,\downarrow}(x',y')$, as

$$U_k(x',y') = U_{k,\uparrow}(x',y') + U_{k,\downarrow}(x',y'), \qquad (4)$$

where $U_{k,\uparrow}(x',y')$ and $U_{k,\downarrow}(x',y')$ are given, respectively, by

$$\begin{split} U_{k,\uparrow}(x',y') &= \sum_{m=1}^{M} \sum_{n=1}^{m} \Lambda_{k,\uparrow}(x' - \tilde{x}_{k,\uparrow,mn},y' \\ &- \tilde{y}_{k,\uparrow,mn}) \Upsilon_{k,\uparrow,mn} \exp(j\Gamma_{k,\uparrow,mn}), \end{split}$$
(5a)

$$\begin{split} U_{k,\downarrow}(x',y') &= \sum_{m=1}^{M-1} \sum_{n=1}^{m} \Lambda_{k,\downarrow}(x' - \tilde{x}_{k,\downarrow,mn},y' \\ &- \tilde{y}_{k,\downarrow,mn}) \Upsilon_{k,\downarrow,mn} \exp(j\Gamma_{k,\downarrow,mn}), \end{split} \tag{5b}$$

where $\tilde{x}_{k,\uparrow,mn}$, $\tilde{y}_{k,\uparrow,mn}$, $\tilde{x}_{k,\downarrow,mn}$, and $\tilde{y}_{k,\downarrow,mn}$ are the dividing points defined, respectively, by

$$\tilde{x}_{k,\uparrow,mn} = (n-1)(x_2 - x_3)/M + (m-n)(x_1 - x_3)/M,$$
(6a)

$$\tilde{y}_{k,\uparrow,mn} = (n-1)(y_2 - y_3)/M + (m-n)(y_1 - y_3)/M, \eqno(6b)$$

$$\tilde{x}_{k,\downarrow,mn} = n(x_2 - x_3)/M + (m - n + 1)(x_1 - x_3)/M,$$

(6c)

$$ilde{y}_{k,\downarrow,mn} = n(y_2 - y_3)/M + (m - n + 1)(y_1 - y_3)/M,$$
(6d)

and the phase and amplitude modulation values at the (m, n)th upward-oriented triangle of the kth



Fig. 2. Diffusive facet surface parameterized by the factor M: (a) upward-oriented triangles, (b) downward-oriented triangles.

facet of the triangle-meshed surface object are denoted $\Gamma_{k,\uparrow,mn}$ and $\Upsilon_{k,\uparrow,mn}$, respectively. Those at the (m,n)th downward-oriented triangle of the kth facet are denoted $\Gamma_{k,\downarrow,mn}$ and $\Upsilon_{k,\downarrow,mn}$, respectively. These phase and amplitude distributions are de-

These phase and amplitude distributions are degrees of freedom of the light field that can be used for expressing specific shading and texture effects of the facet [1,2,9]. Also, these freedom factors can be exploited for improving some properties of the image light fields. For example, by the optimization of the distributions, phase-only CGH with higher transmission efficiency can be synthesized [14,15].

The analytic 2D Fourier transform $A_k(\alpha',\beta')$ of the transmittance function $U_k(x',y')$ of the kth unit triangular facet with a diffusive surface profile is obtained as

$$\begin{split} A_{k}(\alpha',\beta') &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_{k}(x',y') \exp(-j2\pi(\alpha'x' + \beta'y')) dx' dy' \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (U_{k,\uparrow}(x',y') + U_{k,\downarrow}(x',y')) \exp(-j2\pi(\alpha'x' + \beta'y')) dx' dy' \\ &= A_{k,\uparrow}(\alpha',\beta') + A_{k,\downarrow}(\alpha',\beta'), \end{split}$$

where $A_{k,\uparrow}(\alpha',\beta')$ and $A_{k,\downarrow}(\alpha',\beta')$ are given, respectively, by

$$\begin{split} &\sum_{m=1}^{M} \sum_{n=1}^{m} \Upsilon_{k,\uparrow,mn} \exp(j\Gamma_{k,\uparrow,mn}) \exp(-j2\pi(\alpha'\tilde{x}_{k,\uparrow,mn} + \beta'\tilde{y}_{k,\uparrow,mn})) A_{e,k,\uparrow}(\alpha',\beta'), \\ &\sum_{m=1}^{M-1} \sum_{n=1}^{m} \Upsilon_{k,\downarrow,mn} \exp(j\Gamma_{k,\downarrow,mn}) \exp(-j2\pi(\alpha'\tilde{x}_{k,\downarrow,mn} + \beta'\tilde{y}_{k,\downarrow,mn})) A_{e,k,\downarrow}(\alpha',\beta'), \end{split}$$

where $A_{e,k,\uparrow}(\alpha',\beta')$ and $A_{e,k,\downarrow}(\alpha',\beta')$ are the analytic 2D Fourier transforms of the elementary upwardoriented triangular facet $\Lambda_{k,\uparrow}(x',y')$ and the elementary downward-oriented triangular facet $\Lambda_{k,\downarrow}(x',y')$, respectively. The analytic 2D Fourier transform of an arbitrary triangle is derived in the Appendix. In practical computation, computation efficiency can be attained by exploiting the periodicity of similar triangles as manifested in Eq. (7).

The light field distribution in the global coordinate system, $W_k(x, y, z)$, is derived from its local coordinate representation Eq. (3c) with the rotational transform in Eq. (2). Before going into this task, let the Fourier spatial-frequency vector $(\alpha', \beta', \gamma')$ related to the *k*th unit triangular facet be denoted $(\alpha'^{(k)}, \beta'^{(k)}, \gamma'^{(k)})$ with superscript (k). Using Eq. (2), we can see that the components of the Fourier spatial-frequency vector in the

local coordinate system, $\alpha'^{(k)}$, $\beta'^{(k)}$, and $\gamma'^{(k)}$, become the functions of the Fourier spatial-frequency vector (α, β, γ) in the global coordinate system

$$\alpha^{\prime(k)}(\alpha,\beta) = \cos\theta_k \cos\phi_k \alpha + \cos\theta_k \sin\phi_k \beta - \sin\theta_k \gamma,$$
(8a)

$$\beta^{\prime(k)}(\alpha,\beta) = -\sin\phi_k\alpha + \cos\phi_k\beta, \qquad (8b)$$

$$\gamma^{\prime(k)}(lpha,eta) = \sin heta_k \cos \phi_k lpha + \sin heta_k \sin \phi_k eta + \cos heta_k \gamma.$$
(8c)

From Eqs. (8a) and (8b), the relationship among derivatives $d\alpha'^{(k)}(\alpha,\beta)$, $d\beta'^{(k)}(\alpha,\beta)$, $d\alpha$, and $d\beta$ is derived as

$$\begin{bmatrix} d\alpha'^{(k)}(\alpha,\beta) \\ d\beta'^{(k)}(\alpha,\beta) \end{bmatrix} = \begin{pmatrix} \cos\theta_k \cos\phi_k + \frac{\alpha\sin\theta_k}{\gamma} & \cos\theta_k \sin\phi_k + \frac{\beta\sin\theta_k}{\gamma} \\ -\sin\phi_k & \cos\phi_k \end{pmatrix} \times \begin{bmatrix} d\alpha \\ d\beta \end{bmatrix}.$$
(9a)

Hence the differential area in the local coordinate system, $da'^{(k)}d\beta'^{(k)}$ is given by

$$d\alpha^{\prime(k)}(\alpha,\beta)d\beta^{\prime(k)}(\alpha,\beta) = \left|\cos\theta_{k} + \frac{\sin\theta_{k}(\cos\phi_{k}\alpha + \sin\phi_{k}\beta)}{\gamma}\right|d\alpha d\beta. \quad (9b)$$

With Eqs. (8a), (8b), and (9b) substituted into Eq. (3c), the diffraction field in the global coordinate system, $W_k(x, y, z)$, is obtained as

$$\begin{split} W_{k}(x,y,z) &= \eta_{0} \exp(j2\pi(\alpha_{0}x_{k} + \beta_{0}y_{k} + \gamma_{0}z_{k})) \\ &\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{A_{k}(\alpha'^{(k)}(\alpha,\beta) \\ &- \alpha'^{(k)}_{0}(\alpha_{0},\beta_{0}), \beta'^{(k)}(\alpha,\beta) \\ &- \beta'^{(k)}_{0}(\alpha_{0},\beta_{0}), \beta'^{(k)}(\alpha,\beta)) \\ &\times \exp(j2\pi(\alpha(x-x_{k}) + \beta(y-y_{k}) \\ &+ \gamma(z-z_{k})))\} \bigg| \cos \theta_{k} \\ &+ \frac{\sin \theta_{k}(\cos \phi_{k}\alpha + \sin \phi_{k}\beta)}{\gamma} \bigg| d\alpha d\beta, \quad (10) \end{split}$$

where it should be noted that a function $H(\gamma'^{(k)}(\alpha,\beta))$ is included. In the calculation of $W_k(x,y,z)$, the condition of $\gamma'^{(k)} > 0$ must be satisfied; otherwise, the angular spectrum value at this frequency must be zero. For this reason, the unit step function $H(\gamma'^{(k)}(\alpha,\beta))$ is included in Eq. (10), that is, the angular spectrum representation in the global coordinate system. Therefore the analytic angular spectrum of the 3D image light field at z = 0 in the global coordinate system $A_{G,k}(\alpha,\beta)$ takes the form

$$\begin{split} A_{G,k}(\alpha,\beta) &= \eta_0 \exp(j2\pi(\alpha_0 x_k + \beta_0 y_k + \gamma_0 z_k))A_k \\ &\times (\alpha'^{(k)}(\alpha,\beta) - \alpha'^{(k)}_0(\alpha_0,\beta_0), \beta'^{(k)}(\alpha,\beta) \\ &- \beta'^{(k)}_0(\alpha_0,\beta_0))H(\gamma'^{(k)}(\alpha,\beta)) \left|\cos\theta_k \right. \\ &+ \frac{\sin\theta_k(\cos\phi_k\alpha + \sin\phi_k\beta)}{\gamma} \left|\exp(j2\pi[\alpha(-x_k) + \beta(-y_k) + \gamma(-z_k)]). \end{split}$$

This analytic angular spectrum representation formulation gives exact angular spectrum representation in the global coordinate system without the interpolated remapping necessarily used in the previous study [9].

Let us demonstrate the effect of the surface diffusiveness with some simulations. For convenience, the direction vector of the carrier plane wave $(\alpha_0, \beta_0, \gamma_0)$ is set to $(0, 0, 1/\lambda)$ parallel to the *z* axis. The observation angle φ_v is defined by the angle between the z axis (carrier plane wave direction) and the direction vector of an observer's eye from the origin. To understand the effect of the diffusiveness factor M, we inspect the observed 3D holographic images for several values of the observation angle φ_v and the diffusiveness factor M. In Fig. 3, the observed images of three separate triangles with respective diffusive surface profiles parameterized by M = 4 and M =64 are presented (see Eq. (18a)) in Section 4 for the detailed simulation method). It is noted that the phase values at elementary triangles are random. In Figs. 3(a)-3(c), the images of three triangle objects with diffusiveness factor M = 4 observed at specific directions with respective observation angles $\varphi_v = 0^\circ$, $\varphi_v = -1^\circ$, and $\varphi_v = -2^\circ$ are presented, respectively. At the normal direction collinear to the direction of the carrier plane wave, the image with a monocular cue that is focused on the left triangle is observed. In this case, the aperture size of the elementary triangle is relatively so large that the viewing angle of the diffraction field becomes limited to a narrow viewing zone. As seen in Figs. 3(b) and 3(c),



Fig. 3. (Color online) Images of three triangles with the diffusiveness factor of M = 4 observed at observation angles (a) $\varphi_v = 0^\circ$, (b) $\varphi_v = -1^\circ$, (c) $\varphi_v = -2^\circ$. Images of three triangles with the diffusiveness factor of M = 64 observed at observation angles (d) $\varphi_v = 0^\circ$, (e) $\varphi_v = -5^\circ$, (f) $\varphi_v = -10^\circ$.

we can only observe the edges of the triangles because the high-spatial-frequency components that can be observed at the high viewing angle are generated only around the edges of the triangles. However, if we use smaller elementary triangles for composing the three triangle objects by increasing the diffusiveness factor M to 64, the image light field generated by an elementary triangle has a wider angular spectrum bandwidth, so we can observe clearer 3D image of the objects at positions of wide viewing angles. In Figs. 3(d)-3(f), the images of three triangle objects with diffusiveness parameter M = 64 observed at specific directions with respective observation angles $\varphi_v = 0^\circ$, $\varphi_v = -5^\circ$, and $\varphi_v = -10^\circ$ are presented, respectively. At the observation angle $\varphi_v = -10^\circ$, we can observe the facets of tree triangles as well as edges. Therefore, the diffusiveness factor M is the control parameter of the angular spectrum bandwidth of the triangular facet.

3. Image Light Field of Three-Dimensional Objects with Occlusion

In this section, the mathematical representation of image light field radiated from a triangle-meshmodeled 3D surface object with occlusion is described. We can construct the angular spectrum representation of the collective total image light field of the 3D object by simply summing up all partial optical fields generated by all triangles composing a 3D object, which is represented as

$$A^{\text{total}}(\alpha,\beta) = \sum_{k\in\Gamma} A_{G,k}(\alpha,\beta), \quad (12)$$

where Γ indicates the set of facet indices. However, in the above representation, the occlusion effect is not considered. The occlusion indicates that the front facets are observed but the rear parts must be hidden to an observer at a specific observation direction. In Fig. 4(a), the triangle-mesh-based modeling of two hands is shown, where the observation direction is the z direction shown in Fig. 1(a) and the observation longitudinal angle φ_v and azimuthal angle ψ_v are 0°. In Fig. 4(b), a projection image of a dinosaur toward the observation direction, $(\varphi_v, \psi_v) = (20^\circ, 0^\circ)$, is shown. In practice, the dinosaur is properly rotated so that the specified observation direction is matched to the positive z axis and the observer sees the rotated object from the positive z direction.

In these images, the front facets screen the rear facets. In other words, the partial facets appearing in the projection image do contribute to form the holographic image light field for the observer at a specific observation direction. Thus, an algorithm to select the observable facets for a fixed observation direction is necessary. We devise a



Fig. 4. (a) Projection image of triangle-mesh-modeled two hands toward an observation direction $(\varphi_v, \psi_v) = (0^\circ, 0^\circ);$ (b) projection image of triangle-mesh-modeled dinosaur toward an observation direction $(\varphi_v, \psi_v) = (20^\circ, 0^\circ).$

simple geometric observable facet selection method for effectively realizing the occlusion within the triangle-mesh modeling described in Section 2. The two hands and the dinosaur are used as verification examples of experimental reconstruction of CGHs. The two-hands object shown in Fig. 4(a) is composed of 2436 triangles. In our model, a triangular facet has a diffusiveness surface profile with M = 5. In this case, a triangular facet is composed of 15 upward-oriented triangles and 10 downwardoriented triangles. With this setting, the total number of the upward-oriented triangles and that of the downward-oriented-triangles composing the two hands is 36,540 and 24,360, respectively. For 36,540 upward-oriented triangles and 24,360 downwardoriented triangles, we make an upward-oriented triangle lookup table indicator $T_{\uparrow}(\varphi_v, \psi_v, k, m, n)$ and a downward-oriented triangle lookup table indicator $T_{\downarrow}(\varphi_v, \psi_v, k, m, n)$ by using a self-developed ray-tracing program. The lookup table indicator $T_{\uparrow(\downarrow)}(\varphi_v,\psi_v,k,m,n)$ is defined as

$$T_{\uparrow(\downarrow)}(\varphi_v, \psi_v, k, m, n) = \begin{cases} 1 & \text{if the } (m, n) \text{ up } (\text{down}) \text{ triangle of the } k \text{th facet is observable} \\ 0 & \text{if not} \end{cases}$$
(13)

With these indicators prepared, the analytic 2D Fourier transform of the *k*th unit triangular facet in the local coordinate system for a specific observation direction (φ_v, ψ_v) is refined to

$$\begin{split} A_{k}(\alpha',\beta') &= \sum_{m=1}^{M} \sum_{n=1}^{m} T_{\uparrow}(\varphi_{v},\psi_{v},k,m,n)\Upsilon_{k,\uparrow,mn} \\ &\times \exp(j\Gamma_{k,\uparrow,mn}) \\ &\times \exp(-j2\pi(\alpha'\tilde{x}_{k,\uparrow,mn}+\beta'\tilde{y}_{k,\uparrow,mn}))A_{e,k,\uparrow}(\alpha',\beta') \\ &+ \sum_{m=1}^{M-1} \sum_{n=1}^{m} T_{\downarrow}(\varphi_{v},\psi_{v},k,m,n)\Upsilon_{k,\downarrow,mn} \\ &\times \exp(j\Gamma_{k,\downarrow,mn}) \\ &\times \exp(-j2\pi(\alpha'\tilde{x}_{k,\downarrow,mn}+\beta'\tilde{y}_{k,\downarrow,mn}))A_{e,k,\downarrow}(\alpha',\beta') \\ &= A_{k,\uparrow}(\alpha',\beta') + A_{k,\downarrow}(\alpha',\beta'). \end{split}$$

The angular spectrum in the global coordinate system, $A_{G,k}(\alpha,\beta)$ is obtained with Eq. (14) substituted into Eq. (11). The angular spectrum of the collective total image light field $A_G^{\text{total}}(\alpha,\beta)$ is given by Eq. (12). Then the collective total field $W^{\text{total}}(x,y,z)$ is represented by

$$\begin{split} W^{\text{total}}(x,y,z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_{G}^{\text{total}}(\alpha,\beta) \\ &\times \exp(j2\pi(\alpha x + \beta y + \gamma z)) \mathrm{d}\alpha \mathrm{d}\beta. \end{split} \tag{15}$$

Let us apply the addressed geometric occlusion model to the previous three-triangle example. At the observation angles of $\varphi_v = 5^{\circ}$ and $\varphi_v = 10^{\circ}$, the first triangle obstructs the other triangles. In Figs. 5(a) and 5(b), the calculated observed images focused on the left triangle observed with the observation angles $\varphi_v = 5^{\circ}$ and $\varphi_v = 10^{\circ}$ are presented, respectively. In Figs. 5(c) and 5(d), the calculated observed images of three triangles focused on the right triangle observed with the observation angles $\varphi_v = 5^{\circ}$ and $\varphi_v = 10^{\circ}$ are presented, respectively [see Eq. (18a) in Section 4 for the detailed simulation method]. We can see a 3D holographic image of this three-triangle object with a monocular cue and occlusion. In Section 4, CGH reconstruction of the complex 3D objects shown in Fig. 4 will be demonstrated.

The occlusion method used in this paper is similar to the geometric method with the point-source model. In our method, a small triangle is analogous to a point in the point-source model. It is assumed that the CGH synthesized by the developed model is encoded into dynamic SLM and that its angular spectrum bandwidth is narrow to be appropriate for a single view. The geometric occlusion method creates projection image toward a specific direction with a monocular cue, i.e., a defocus effect. The limitation in the narrow viewing angle can be systematically overcome by the creation of multiview using a tiled SLM array. If we have a tiled SLM array, each component SLM of the SLM array generates a respective projection image with a monocular cue.



Fig. 5. (Color online) Images of three triangles, focused on the left-hand triangle, observed at the observation angles (a) $\varphi_v = 5^{\circ}$ and (b) $\varphi_v = 10^{\circ}$. Images of three triangles focused on the right-hand triangle, observed at the observation angles (c) $\varphi_v = 5^{\circ}$, (d) $\varphi_v = 10^{\circ}$.

Although the occlusion method in this paper is appropriate for a specific direction view, recently a more rigorous wave-oriented occlusion method was proposed [16]. The wave-oriented occlusion algorithm in [16] addresses the wave occlusion effect by a tilted facet using the fast Fourier transform. In the context of this paper, it would be an important problem to combine the introduced analytic angular spectrum formulation into the wave-oriented occlusion approach.

4. Synthesis and Reconstruction of Computer-Generated Holograms

In this section, synthesis of CGH based on the developed mathematical model and reconstruction are demonstrated. Actually, we can use the phase profile of the angular spectrum of the 3D image light field, $A_G^{\text{total}}(\alpha, \beta)$, as a CGH-producing 3D holographic image light field. It is encoded into the phase SLM by the relationship

$$\Pi(u,v) = A_G^{\text{total}}\left(\frac{u}{\lambda f}, \frac{v}{\lambda f}\right),$$
(16)

where f is the focal length of the Fourier transform lens used in CGH reconstruction. The experimental setup for CGH reconstruction and observation is shown in Fig. 6. In the ideal reconstruction, the SLM should exactly generate the amplitude profile as well as the phase profile of $\Pi(u, v)$. In this case, the light field distribution at the phase SLM is given by

$$F(u,v) = \frac{1}{j\lambda f} \Pi(u,v) = \frac{1}{j\lambda f} \sum_{k} A_{G,k} \left(\frac{u}{\lambda f}, \frac{v}{\lambda f}\right).$$
(17)



Fig. 6. Schematic of CGH reconstruction and observation.

The light field distribution by the singlet with a focal length of f is represented by the Fresnel transform

$$\begin{split} \tilde{W}_{\rm rec}(x,y;\Delta z) = & \frac{1}{j\lambda f} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi(u,v) \\ & \times \exp\left(-\frac{j\pi}{\lambda f} \left[\left(\frac{\Delta z}{f}\right) (u^2 + v^2) \right. \\ & \left. - 2(ux + vy) \right] \right] du dv. \end{split}$$
(18a)

At the lens focal plane $\Delta z = 0$, the light field distribution is given by

$$\begin{split} \tilde{W}_{\rm rec}(x,y;0) &= -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_G^{\rm total}\left(\frac{u}{\lambda f},\frac{v}{\lambda f}\right) \\ &\times \exp\left(j2\pi\left(x\frac{{\rm d}u}{\lambda f}+y\frac{{\rm d}v}{\lambda f}\right)\right)\left(\frac{{\rm d}u}{\lambda f}\right)\left(\frac{{\rm d}v}{\lambda f}\right) \\ &= -W_{\rm total}(x,y). \end{split}$$
(18b)

Therefore, under ideal conditions, the reconstructed light field is exactly the same as the original light field.

However, in practice, the phase profile of the angular spectrum $A_G^{\text{total}}(\alpha,\beta)$ is usually encoded into the phase SLM. Although the amplitude modulation profile is functionally dependent on the transmission characteristics of the used phase SLM, we can assume that the amplitude transmission is approximately a constant. In this case, the reconstructed light field is represented by the Fresnel transform of the CGH:

$$\bar{W}_{\rm rec}(x,y;\Delta z) = \frac{1}{j\lambda f} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(j\measuredangle \Pi(u,v)) \\ \times \exp\left(-\frac{j\pi}{\lambda f} \left[\left(\frac{\Delta z}{f}\right)(u^2 + v^2) - 2(ux + vy)\right]\right) du dv.$$
(19)

Since the amplitude information is discarded, practical reconstruction using the phase SLM can induce noise in the observed images.

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In the experiment, an incident He–Ne laser beam with wavelength of 632.8 nm modulated by the phase SLM (Epson L3P06, pixel size $12 \mu m$) with holographic information is optically Fourier transformed by a singlet with focal length of f (= 20 cm). With a CCD camera (Kodak MegaPlus ES 1.0, pixel size $9\,\mu$ m), the image light fields are observed at various defocus positions. In Figs. 7 and 8, numerical reconstruction using Eqs. (18a) and (18b) and experimental reconstruction corresponding to Eq. (19) are presented for validating the developed mathematical model and confirming the monocular cue, i.e., defocus effect, respectively. In the case of the two hands, the surface diffusiveness factor M is set to 5. The phase distributions $(\Gamma_{k,\uparrow,mn} \text{ and } \Gamma_{k,\downarrow,mn})$ are a random phase distribution, and the amplitude distributions $(\Upsilon_{k,\uparrow,mn} \text{ and } \Upsilon_{k,\downarrow,mn})$ are designed to produce a shade effect as shown in Fig. 4(a). The amplitude and phase distributions of the angular spectrum are shown in Figs. 7(a) and 7(b), respectively. The size of the CGH is 501×501 . In Figs. 7(c) and 7(e), the simulation results of two cases focused on the right hand and the left hand are presented, respectively. In Figs. 7(d) and 7(f), experimental results corresponding to Figs. 7(c) and 7(e) are shown, respectively. The hard clip of the amplitude profile



Fig. 7. (Color online) CGH of two hands: (a) amplitude profile $|\Pi(u,v)|$, (b) phase profile $\measuredangle \Pi(u,v)$. Image of two hands focused on the left hand: (c) simulation and (d) experiment. Image of two hands focused on the right hand: (e) simulation and (f) experiment.

to a constant induces some background noise and the center bright dc spot in the reconstructed image light field. We can see that the synthesized CGH produces a volumetric 3D holographic image light fields with clear defocus effect between two separated 3D objects. In the case of the dinosaur, the surface diffusiveness factor M is set to 4. The phase distributions $(\Gamma_{k,\uparrow,mn}$ and $\Gamma_{k,\downarrow,mn})$ are also a random phase distribution, and the amplitude distributions $(\Upsilon_{k,\uparrow,mn} \text{ and } \Upsilon_{k,\downarrow,mn})$ are determined to represent a shade effect as shown in Fig. 4(b). In Figs. 8(a) and 8(b), the amplitude and phase distributions of the angular spectrum are shown, respectively. The size of the CGH is 501×501 . In Figs. 8(c) and 8(e), simulation results of two cases focused on the head and the tail are presented, respectively. In Figs. 8(d) and 8(f), experimental results corresponding to Figs. 8(c) and 8(e), respectively, are shown. By this example, we show that the synthesized CGH produces a single volumetric 3D holographic image light field with clear defocus effect within a single volume object.

The described mathematical modeling is a successful model for describing 3D image light fields emitted from general 3D surface objects with occlusion. The developed model can be extensively used for digital holography, in particular, CGH synthesis. The wave-oriented CGH synthesis method is different from the projection-type CGH [17,18] in that the projection-type CGH is just showing an image projected onto each viewing direction without a defocus effect, but the wave-oriented CGH synthesis method calculates the real physical light field with a monocular cue.

5. Conclusion

In conclusion, a mathematical model of trianglemesh-modeled 3D surface objects for digital holography has been developed. The proposed mathematical model includes the analytic angular spectrum representation of image light fields emitted from 3D objects with occlusion and a diffusive surface model feasible for producing surface texture and shade effects. The developed model can be extensively used for digital holography as well as CGH synthesis.



Fig. 8. (Color online) CGH of a dinosaur: (a) amplitude profile $|\Pi(u,v)|$ and (b) phase profile $\measuredangle\Pi(u,v)$. Image of the dinosaur focused on the head: (c) simulation and (d) experiment. Image of the dinosaur focused on the tail: (e) simulation and (f) experiment.

Appendix: Two-Dimensional Fourier Transform of a Triangle

Here, the 2D Fourier transform of a triangle is derived. Let us consider a triangle *ABC* shown in Fig. 9(a). Point D is the perpendicular foot. To obtain the Fourier transform of this triangle, first the triangle *ABC* is shifted to the triangle *A'B'C'* such that point D is put on the origin as shown in Fig. 9(b). Then, the triangle *A'B'C'* is rotated by the angle of ψ to the triangle *A'B'C''* such that the line *A''B''* is on the *x* axis as shown in Fig. 9(c).

The coordinate of point D, (x_c, y_c) , is given by

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \frac{1}{(y_2 - y_1)^2 + (x_2 - x_1)^2} \begin{pmatrix} (x_3 - x_1)(x_2 - x_1)(y_2 - y_1) + y_3(y_2 - y_1)^2 + y_1(x_2 - x_1)^2 \\ (y_3 - y_1)(x_2 - x_1)(y_2 - y_1) + x_3(x_2 - x_1)^2 + x_1(y_2 - y_1)^2 \end{pmatrix},$$
(A1)

and $(\cos(\psi), \sin(\psi))$ is given by

$$(\cos(\psi), \sin(\psi)) = (y_3 - y_c, x_3 - x_c) / \sqrt{(x_3 - x_c)^2 + (y_3 - y_c)^2} \,. \tag{A2}$$





Fig. 9. Fourier transform of a triangle by simple geometric transform.

Points A", B", and C" are denoted (-a, 0), (b, 0), and (0, c), respectively, where a, b, c are given by

$$(a, b, c) = (-(x_1 - x_c) \cos \psi + (y_1 - y_c) \sin \psi, (x_2 - x_c) \cos \psi - (y_2 - y_c) \sin \psi, (x_3 - x_c) \sin \psi + (y_3 - y_c) \cos \psi).$$
(A3)

Let the Fourier transform of the triangle A''B''C''be denoted $F(f''_x, f''_y)$. The Fourier transform $F(f''_x, f''_y)$ is given by Let the Fourier transform of the triangle ABC be denoted $\bar{F}(f_x, f_y)$; then $\bar{F}(f_x, f_y)$ is obtained by

$$\bar{F}(f_x, f_y) = F(f_x \cos \psi - f_y \sin \psi, f_x \sin \psi + f_y \cos \psi) \exp(-j2\pi (f_x x_c + f_y y_c)).$$
(A5)

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$$\begin{split} F(f''_{x},f''_{y}) \\ &= \begin{cases} \left(\frac{-jc}{2\pi f''_{x}}\right) \{-e^{-j2\pi f''_{x}b}e^{j\pi (f''_{x}b-f''_{y}c)}\operatorname{sinc}(f''_{x}b-f''_{y}c) + e^{j2\pi f''_{x}a}e^{j\pi (f''_{x}a+f''_{y}c)}\operatorname{sinc}(f''_{x}a+f''_{y}c)\} & \text{for } f''_{x} \neq 0 \\ \left(\frac{a+b}{c}\right)e^{-i2\pi f''_{y}c}\left\{\frac{-1-(j2\pi c f''_{y}-1)e^{j2\pi f''_{y}c}}{(2\pi f''_{y})^{2}}\right\} & \text{for } f''_{x} = 0, f''_{y} \neq 0 \end{cases} \quad (A4) \\ \frac{(a+b)c}{2} & \text{for } f''_{x} = 0, f''_{y} = 0 \end{split}$$

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