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Iterative Fourier Transform Algorithm with Adaptive Regularization Parameter Distribution for Optimal Design of Diffractive Optical Elements

Hwi KIM and Byoungho LEE*

School of Electrical Engineering, Seoul National University, Kwanak-Gu Shinlim-Dong, Seoul 151-744, Korea

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Generally, uniformity can be improved at the cost of a low diffraction efficiency in the design of diffractive optical elements by the conventional iterative Fourier transform algorithm. This trade-off is a fundamental limitation of the conventional iterative Fourier transform algorithm. However in this letter, a novel iterative Fourier transform algorithm with adaptive regularization parameter distribution for alleviating the trade-off between uniformity and diffraction efficiency and for improving uniformity is devised on the basis of the theoretical analysis of the trade-off property in the conventional iterative Fourier transform algorithm scheme. The validity of the proposed algorithm is proved by numerical experiments.
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KEYWORDS: diffractive optical element, iterative Fourier transform algorithm, regularization parameter, uniformity

A diffractive optical element (DOE) is a device that can form a specific diffraction image on an image plane. The optimal design of a DOE is realized by solving the phase retrieval problem, which is an inverse problem of constructing the phase profile of the DOE from the knowledge of the object diffraction image on the image plane and the incident beam profile on the DOE plane. In the scalar and paraxial domains, many efficient methods for obtaining quasi-optimal solutions to the phase retrieval problem have been investigated.[1–7] The most preferred algorithms among them are the iterative Fourier transform algorithm (IFTA) and its modified forms. It is noteworthy that the formulation of IFTA by Kotlyar *et al.* is a refined generalization of the IFTA, which can cover almost all historic variants of IFTA and include key concepts of IFTA as parameterization by the use of a relaxation parameter to control the convergence rate, the excess degree of freedom of exploiting amplitude and phase freedom, and regularization.[1] In ref. 1, the generating functional of IFTA and the derivation of IFTA based on the variational method are described. The convergence of IFTA is discussed in ref. 2.

The evaluation features of the solution, that is, the phase profile of the DOE, are mean square error (MSE), diffraction efficiency (DE), the uniformity of the diffraction intensity distribution, and signal-to-noise ratio (SNR). Particularly, the uniformity of the diffraction intensity distribution indicates the flatness of the signal distribution in the image plane and is defined as

$$\text{Uniformity} = (|F|_{\max} - |F|_{\min})/(|F|_{\max} + |F|_{\min}), \quad (1)$$

where F denotes the diffraction signal. The theoretical investigation of the relationship between several evaluation features and the convergence characteristics of IFTA is mathematicaly difficult because of its inherent nonlineaity, that is, the hard clip operation in the DOE plane that abruptly replaces the amplitude of the complex signal with that of the pregiven incident signal profile. Therefore, several characteristics of IFTA are known only empirically. Many variants of IFTA have been developed huristically.[6,7]

One of the common understandings of IFTA is the trade-off between uniformity and diffraction efficiency. Generally, uniformity can be improved at the cost of a low diffraction efficiency in the design of DOEs by methods such as IFTA. This trade-off is a fundamental limitation of IFTA. The theoretical study on the trade-off is presumed to be an important issue in the design of DOEs. Furthermore, to alleviate the trade-off between uniformity and diffraction efficiency is a target of theoretical study on IFTA. However, the reason for the existence of such a trade-off has not been substantiated and furthermore an appropriate method to alleviate the trade-off has not yet materialized.

In this letter, a novel method of alleviating the trade-off is investigated through the theoretical analysis of the trade-off between the uniformity and diffraction efficiency of the conventional IFTA scheme. According to the proposed novel IFTA scheme, the trade-off is lightened, which can improve uniformity particularly. The validity of the proposed method will be proved by numerical experiments.

Initially, we construct a generating functional of the classical IFTA as

$$E(F) = \iint_{-\infty}^{\infty} |D_{\mathrm{B}}|F| - F_0|^2 \mathrm{d}x\mathrm{d}y + \alpha \iint_{S} |F|^2 \mathrm{d}x\mathrm{d}y, \quad (2)$$

where D_{B} is the area-limiting operator defined as

$$D_{\mathrm{B}}|F| = \begin{cases} |F| & (x, y) \in S \\ 0 & (x, y) \notin S \end{cases}, \quad (3)$$

where F is the calculated complex light signal, F_0 is the objective positive real signal amplitude, and S is the signal area.[3] The minimization of the generating functional eq. (2) leads to the acquisition of the phase profile of the DOE, forming the complex light signal F, that is, the diffraction image in the image plane. However the minimization of the second term of eq. (2) induces a decrease in diffraction efficiency. Later, it will be clarified that the second integral of eq. (2) plays a role in balancing the trade-off between the uniformity and diffraction efficiency of the diffraction image through the regularization parameter α. An iterative algorithm can be derived to minimize the functional eq. (2) by a variational method, the Landweber iteration method. The variation of the functional eq. (2) is given by

*Corresponding author. E-mail address: byoungho@snu.ac.kr

$$\delta E(F) = E(F+\delta F) - E(F) = 2\iint_{-\infty}^{\infty} \mathrm{Re}[-(|F_0|\exp(i\psi) + D_B F + D_B \alpha F)\delta F^*]\mathrm{d}x\mathrm{d}y, \tag{4}$$

where Re[·] is the real part of the complex number. The variation of F is determined to generate the maximum negative variation of δE, then δF takes the form

$$\delta F = \bar{F} - F = -\lambda[D_\mathrm{B}(1+\alpha)F - F_0\exp(j\psi)], \tag{5}$$

where λ is the relaxation parameter and ψ is the phase distribution of F. Hence the nth iterate modified in the image plane takes the form

$$\bar{F}_n = F_n - \lambda[D_B(1+\alpha)F_n - F_0\exp(j\psi_n)]. \tag{6}$$

Then the $(n+1)$th light signal F_{n+1} is obtained by applying the error-reduction operator to eq. (6) as

$$F_{n+1} = FrD_\mathrm{A}Fr^{-1}(\bar{F}_n), \tag{7}$$

where operator D_A expresses the surface boundary condition in the DOE plane as

$$D_\mathrm{A}G = \begin{cases} A_0\exp[j\arg(G)] & (u,v)\in\Omega \\ 0 & (u,v)\notin\Omega \end{cases}, \tag{8}$$

where Ω denotes the encoding area in the DOE plane, $\arg(G)$ is the phase function of the complex function G, and Fr denotes the Fresnel transform. Equation (7) causes the signal F_{n+1} fulfill the constraints in the DOE plane. Equations (6) and (7) describe the IFTA for the design of a DOE and eq. (6) can be rewritten as

$$\bar{F}_n = \begin{cases} \lambda F_0\exp(j\psi_n) + (1-\lambda-\lambda\alpha)F_n & (x,y)\in S \\ F_n & (x,y)\notin S \end{cases}. \tag{9}$$

The phase profile of the DOE can be obtained as the phase term of

$$G_{n+1} = D_\mathrm{A}Fr^{-1}(\bar{F}_n). \tag{10}$$

Next, the objective functional eq. (2) can be rearranged with a little manipulation as

$$E(F) = (\alpha+1)\iint_S \left[|F| - \frac{F_0}{(\alpha+1)}\right]^2 \mathrm{d}x\mathrm{d}y + \frac{\alpha}{(\alpha+1)}\iint_S F_0^2\mathrm{d}x\mathrm{d}y. \tag{11}$$

The newly arranged functional eq. (11) is the sum of two terms. The first term is considered as the MSE of the signal distribution F with respect to the objective signal scaled by $(\alpha+1)^{-1}$. The second term indicates the total energy of the incident optical wave multiplied by a constant $\alpha(\alpha+1)^{-1}$. We observe that the weight factor of the first integral is $(\alpha+1)$. Therefore as α increases, the weight factor of MSE (between the signal distribution and the scaled objective signal function) increases and IFTA will make an attempt to further minimize the first term. The minimization of this MSE will lead to an improvement in the uniformity of the signal distribution in the signal area since the even distribution of the signal will further decrease the MSE. However, as stated previously, the increase in the regularization parameter induces the decrease in diffraction efficiency.

It is noted that the minimization of only MSE does not guarantee the best uniformity of the diffraction signal since functional structures of MSE and uniformity are correlated weakly and their mathematical properties are so different. There exists a possibility that the MSE of the case, in which all the samples of F except one sample match the target values exactly and the excluded sample has a relatively big mismatch from the target value, can be smaller than that of the case in which all the samples of F have a relatively small mismatch from the corresponding target values. But the uniformity of the former case is worse than that of the latter case. On the other hand, it is definite that the optimization of only the uniformity will not confirm the reduction in the MSE value. That is, uniformity and MSE are only weakly correlated in the conventional IFTA scheme with a constant regularization parameter. Also, there is a trade-off between uniformity and diffraction efficiency as stated previously.

However, we contemplated a method of strongly correlating uniformity with MSE and worked out a method of alleviating the trade-off between the uniformity and diffraction efficiency of IFTA. We consider the regularization parameter as a distribution $\alpha(x,y)$ not a constant value. Therefore the generating functional eq. (11) is modified to the form

$$E(F) = \iint_S (\alpha+1)\left[|F| - \frac{F_0}{(\alpha+1)}\right]^2 \mathrm{d}x\mathrm{d}y + \iint_S \frac{\alpha}{(\alpha+1)} F_0^2\mathrm{d}x\mathrm{d}y. \tag{12}$$

Let us define the distribution of the regularization parameter $\alpha(x,y)$ as

$$\alpha(x,y) = \frac{2\gamma}{\pi}\tan^{-1}\left(\frac{|F(x,y)| - F_0(x,y)}{F_0(x,y)}\right) + \gamma - 1 \tag{13}$$

where γ is a tuning parameter. The form of eq. (13) is devised for $\alpha(x,y)$ to satisfy the inequality condition

$$\alpha(x,y) + 1 > 0 \tag{14}$$

which is a necessary condition to obtain a correct least-squares formulation as noted in eq. (12). The regularization parameter distribution $\alpha(x,y)$ is dependent on $|F(x,y)| - F_0(x,y)$ and influences the integrand of the first integral in eq. (12). When the value of $||F(x',y')| - F_0(x',y')|$ at a point (x',y') increases, the term of the integrand of the first integral $\{|F(x',y')| - F_0(x',y')/[\alpha(x',y')+1]\}^2$ in eq. (12) becomes larger, irrespective of the sign of the value of α. Since the term $\{|F(x',y')| - F_0(x',y')/[\alpha(x',y')+1]\}^2$ increases faster than $[\alpha(x',y')+1]$ decreases or increases for the variation of $\alpha(x',y')$, the integrand of eq. (12) $[\alpha(x',y')+1]\{|F(x',y')| - F_0(x',y')/[\alpha(x',y')+1]\}^2$ always increases as the value of $||F(x',y')| - F_0(x',y')|$ increases. Therefore the weighting assigned to a certain point (x',y') in the evaluation of MSE, the integrand of the first integral of eq. (12), is magnified at the rate proportional to the value of $||F(x',y')| - F_0(x',y')|$. The distribution of the value of $||F(x',y')| - F_0(x',y')|$ indicates the uniformity of the diffraction image in the image plane. Implicitly, uniformity and MSE are correlated strongly through the distribution of the regularization parameter (13). We call eq. (13) the adaptive regularization parameter distribution (ARPD). The trade-off between diffraction efficiency and uniformity is balanced through the tuning parameter γ in the proposed scheme (eqs. (12) and (13)).

The ARPD is applied in the manipulation of eq. (9) by substituting eq. (13) into eq. (9) as

$$\bar{F}_n = \begin{cases} \lambda F_0 \exp(j\psi_n) + \left(1 - \lambda - \lambda\left(\dfrac{2\gamma}{\pi}\tan^{-1}\left[\dfrac{(|F(x,y)| - F_0(x,y))}{F_0(x,y)}\right] + \gamma - 1\right)\right)F_n & (x,y) \in S \\ F_n & (x,y) \notin S \end{cases}. \quad (15)$$

In the proposed IFTA scheme, an appropriate relaxation parameter λ and a tuning parameter γ are determined experimentally.

To compare the performace of the proposed and the conventional IFTA schemes, the phase profiles of the DOEs and the diffraction images are obtained by the conventional and the proposed IFTAs for several values of regularization parameter α and tuning parameter γ, respectively. The object diffraction image is selected as a rectangular image and the radius of DOE is taken as 0.5 mm. The size of the computation grid is 64×64 and the regularization parameter λ is taken as 1. The values of uniformity for the conventional and the proposed IFTAs are compared with the same diffraction efficiency in the range from 77% to 92% in Fig. 1. The diffraction efficiency can be controlled by varying regularization parameter α for the conventional IFTA scheme and tuning parameter γ for the proposed IFTA scheme. The result in Fig. 1 shows that the uniformity of the proposed IFTA is superior to that of the conventional IFTA with the same diffraction efficiency in all ranges. Particularly, Figs. 2(a) and 3(a) show, respectively, the intensity distribution of the diffraction image obtained by the conventional IFTA and that by the proposed IFTA, which have almost the same diffraction efficiency of approximately 84.5%. However, by comparing Figs. 3(a) and 2(a) it is

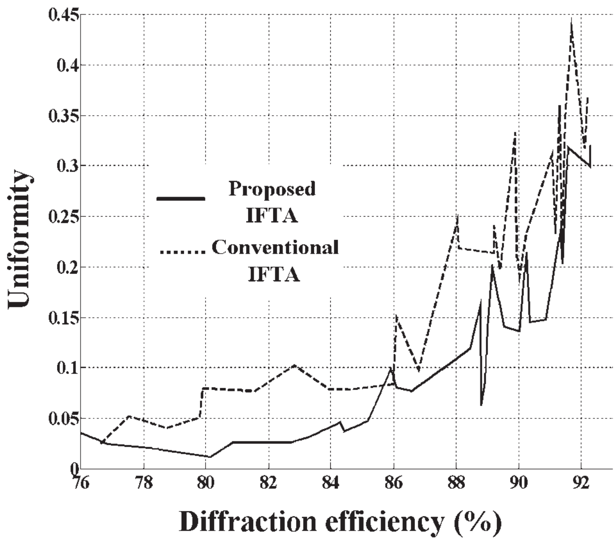

Fig. 1. Comparison of the performance of the conventional and the proposed IFTAs with respect to the trade-off between diffraction efficiency and uniformity. The number of iterations was 100 times for both the conventional and proposed IFTAs.

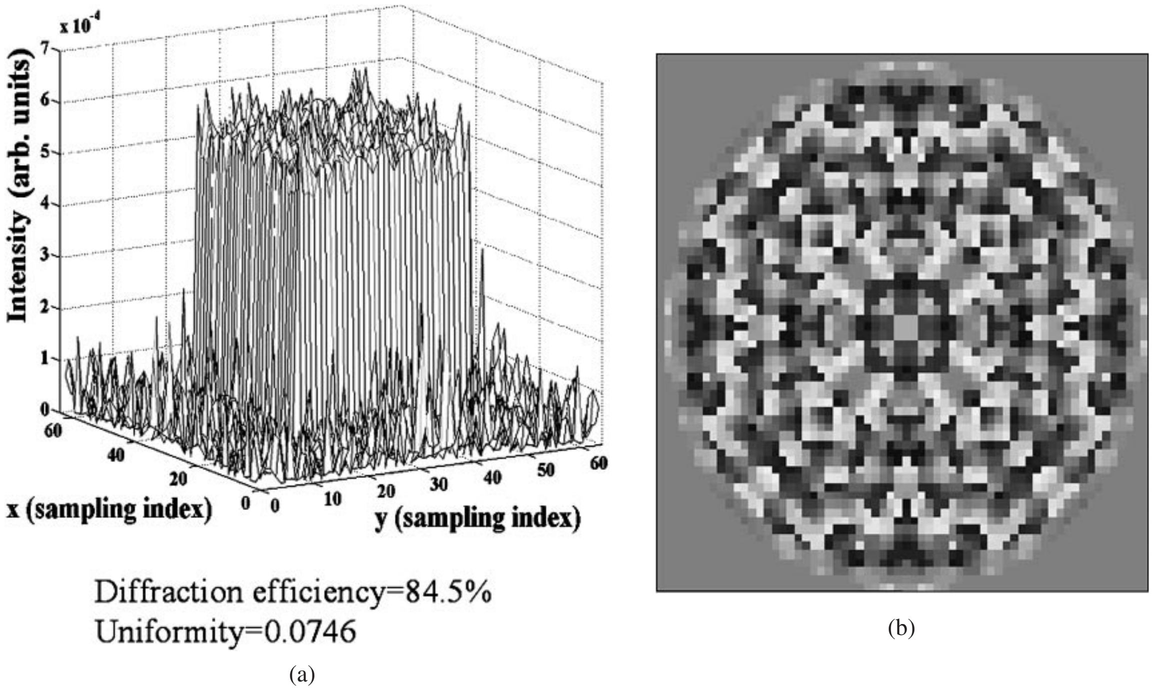

Fig. 2. (a) Intensity distribution of the diffraction image generated by the DOE with (b) the phase profile obtained using the conventional IFTA.

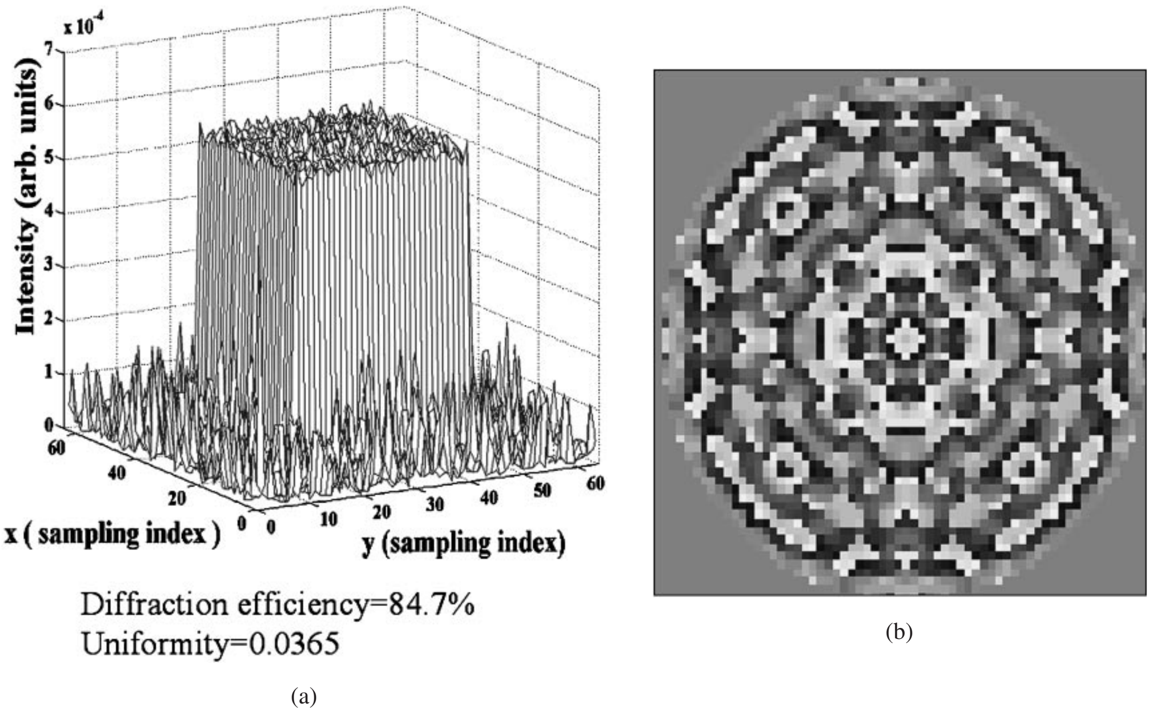

Fig. 3. (a) Intensity distribution of the diffraction image generated by the DOE with (b) the phase profile obtained using the proposed IFTA.

evident that the diffraction image of the proposed IFTA has a better uniformity than that of the conventional IFTA. The former has a uniformity of 0.0365 and the latter has a uniformity of 0.0746. Therefore it is proved that the ARPD alleviates the trade-off between uniformity and diffraction efficiency and the proposed IFTA with APRD improves uniformity markedly, compared with the conventional IFTA.

In conclusion, in this letter, a novel IFTA scheme with ARPD was proposed. It was confirmed theoretically and experimentally that the proposed IFTA has better performance than that of the conventional IFTA. The DOE designed by the proposed IFTA scheme was proven to form a diffraction image with a better uniformity at the same diffraction efficiency than the conventional IFTA scheme.

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